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Painlevé Analysis and Integrability Properties of a 2+1 Nonrelativistic Field Theory

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Abstract

A model for planar phenomena introduced by Jackiw and Pi and described by a Lagrangian including a Chern-Simons term is considered. The associated equations of motion, among which a 2+1 gauged nonlinear Schrödinger equation, are rewritten into a gauge independent form involving the modulus of the matter field. Application of a Painlevé analysis, as adapted to partial differential equations by Weiss, Tabor and Carnevale, shows up resonance values that are all integer. However, compatibility conditions need be considered which cannot be satisfied consistently in general. Such a result suggests that the examined equations are not integrable, but provides tools for the investigation of the integrability of different reductions. This in particular puts forward the familiar integrable Liouville and 1+1 nonlinear Schrödinger equations.

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1 Introduction

Field theories involving Chern-Simons terms have been thought to play a role in the description of planar phenomena, among which the fractional quantum Hall effect and the high- T_c superconductivity [1].

Here we are concerned with a model introduced by Jackiw and Pi [2] which describes a self-interacting non-relativistic matter field $\psi \equiv \psi(x_\nu)$, $\nu = 0, 1, 2$, coupled to an abelian gauge field $A_\mu \equiv A_\mu(x_\nu)$, $\mu, \nu = 0, 1, 2$. The classical equation of motion for ψ is a gauged 2+1 nonlinear Schrödinger equation

$$iD_0\psi + \frac{1}{2m}\mathbf{D}^2\psi + g(\psi^*\psi)\psi = 0 \quad (1a)$$

with

$$D_0 \equiv \partial_0 - iA_0, \quad \mathbf{D} \equiv \nabla - i\mathbf{A}.$$

In the corresponding Lagrangian, the standard Maxwell kinetic term for the gauge field is replaced by a Chern-Simons term [3]

$$-\frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho,$$

so that Eq.(1a) is supplemented by ($i, j = 1, 2$)

$$\psi^*\psi - \kappa\epsilon_{ij}\partial_i A_j = 0, \quad (1b)$$

$$J_i - \kappa\epsilon_{ij}(\partial_j A_0 - \partial_0 A_j) = 0, \quad (1c)$$

with

$$J_i \equiv \frac{i}{2m}[(D_i\psi)^*\psi - \psi^*(D_i\psi)]$$

(the equations dealt with in [2] are recovered by setting $A_0 \rightarrow -A_0$, $\kappa \rightarrow -\kappa$). The fields A_μ are indeed purely auxiliary: upon fixing the gauge, they can be expressed in terms of $\psi^*\psi$ and of the current \mathbf{J} . Eq.(1a) then becomes a non-linear integro-differential equation for ψ . Another peculiarity of the model lies in the fact that exact solutions associated with *static* self dual (or anti self dual) configurations can be constructed for $mg|\kappa| = 1$ [2].

However, important questions still remain unanswered concerning the integrability properties of this model. In this respect, a Painlevé analysis in a form adapted to partial differential equations by Weiss, Tabor and Carnevale (WTC) [4] may be a source of information.

Such an analysis is performed below in Section 2. It gives evidence that Eqs.(1) are *not* integrable, although they naturally admit integrable reductions among which the familiar Liouville and 1+1 nonlinear Schrödinger equations.

2 The Painlevé-WTC analysis

To simplify matters, it is first worthwhile to introduce dimensionless fields and coordinates defined by

$$\psi = |\kappa|^{\frac{3}{2}} r e^{i\chi}, \quad A_0 = \frac{\kappa^2}{2m} w + \partial_0 \chi, \quad A_1 = \kappa u + \partial_1 \chi, \quad A_2 = \kappa v + \partial_2 \chi,$$

and by

$$x_0 = \frac{2m}{\kappa|\kappa|} t, \quad x_1 = \frac{x}{|\kappa|}, \quad x_2 = \frac{y}{|\kappa|}.$$

The new (real) fields $u \equiv u(x, y, t)$, $v \equiv v(x, y, t)$, $w \equiv w(x, y, t)$, $r \equiv r(x, y, t)$ satisfy equations readily deduced from (1). For further convenience, these are ordered as

$$u_y - v_x = -r^2, \tag{2a}$$

$$u_t - w_x = -2vr^2, \tag{2b}$$

$$v_t - w_y = 2ur^2, \tag{2c}$$

$$r_{xx} + r_{yy} = 2Cr^3 - r(w - u^2 - v^2), \tag{2d}$$

$$(r^2)_t = 2(ur^2)_x + 2(vr^2)_y, \tag{2e}$$

with $C \equiv 1/B = -mg|\kappa|$ and, as usual, $u_y = \partial_y u$, $v_x = \partial_x v$, etc.

Note that in the static limit (no time dependence), with moreover $C^2 \equiv 1/B^2 = 1$, Eqs.(2) are associated with the configurations considered in [2]. In this case, they admit the particular solutions $u = -B(r_y/r)$, $v = B(r_x/r)$, $w = Br^2$, with r satisfying the 2+0 Liouville equation

$$\mathcal{L}(r) \equiv \left(\frac{r_x}{r}\right)_x + \left(\frac{r_y}{r}\right)_y - Cr^2 = 0 \tag{3}$$

(indeed the physical, zero energy, solutions are obtained for $C \equiv 1/B = -1$).

On the other hand, if all functions in Eqs.(2) only depend on x, y through the combination $\xi = x + \gamma_c y$, with γ_c a constant that may be zero, then we can introduce $\theta \equiv \theta(\xi, t)$ such that $\theta_\xi = (u + \gamma_c v)/(1 + \gamma_c^2)$, $\theta_t = w - u^2 - v^2 + (u + \gamma_c v)^2/(1 + \gamma_c^2)$. In addition, $\Psi(\xi, t) \equiv r(\xi, t) \exp i\theta(\xi, t)$ can be shown to satisfy the 1+1 nonlinear Schrödinger equation (NLSE)

$$-i\Psi_t + (1 + \gamma_c^2)\Psi_{\xi\xi} = 2C|\Psi|^2\Psi. \tag{4}$$

Let us also emphasize that the phase χ of ψ , which is closely related to the gauge invariance properties of (1), no longer appears in the *gauge independent* system (2). Furthermore, (2e) is nothing else than the compatibility condition of (2a-c) and amounts to the continuity equation $\partial_0(\psi^*\psi) + \nabla \cdot \mathbf{J} = 0$ associated with (1).

Therefore, any Painlevé WTC analysis of (1) may be restricted to (2a-d). At the same time, it is interesting to examine whether this allows the afore mentioned reductions (3), (4) to be put forward.

As in [4], we perform this analysis by looking for solutions u, v, w, r whose behaviour about any movable non characteristic singular manifold $\phi \equiv \phi(x, y, t) = 0$ is defined by generalized Laurent series

$$u = \sum_{j=0}^{\infty} u_j \phi^{j-p_1}, \quad v = \sum_{j=0}^{\infty} v_j \phi^{j-p_2}, \quad w = \sum_{j=0}^{\infty} w_j \phi^{j-p_3}, \quad r = \sum_{j=0}^{\infty} r_j \phi^{j-p_4}, \quad (5a)$$

with *a priori* unknown leading powers p_1, p_2, p_3, p_4 . As usual, we suppose that the coefficients $u_j \equiv u_j(x, y, t)$, $v_j \equiv v_j(x, y, t)$, $w_j \equiv w_j(x, y, t)$, $r_j \equiv r_j(x, y, t)$ only depend on the derivatives of ϕ , with e.g. $\phi_x \neq 0$ when $\phi = 0$. Moreover, we assume that the relevant relations satisfied by these coefficients are merely obtained by inserting (5a) into (2) and by balancing the contributions at each order of ϕ in the resulting equations

$$\sum_{k=0}^{\infty} E_k^{(i)} \phi^{k-q_i} = 0, \quad (5b)$$

$i = 1, 2, 3, 4$ for (2a,b,c,d).

Such operations, when applied to the dominant terms corresponding to $k = 0$, naturally put forward solutions that are *a priori* singular about $\phi = 0$, with

$$p_1 = p_2 = p_4 = 1, \quad p_3 = 2$$

in (5a), and $q_1 = 2, q_2 = q_3 = q_4 = 3$ in (5b). At the same time, they give

$$u_0 = B\phi_y = B\gamma\phi_x, \quad v_0 = -B\phi_x, \quad w_0 = B^2(\phi_x^2 + \phi_y^2) = B^2(1 + \gamma^2)\phi_x^2 \quad (6a, b, c)$$

and

$$r_0^2 = B(\phi_x^2 + \phi_y^2) = B(1 + \gamma^2)\phi_x^2. \quad (6d)$$

In these expressions, γ stands for the quantity

$$\gamma = \frac{\phi_y}{\phi_x} \quad (7)$$

which is invariant under the Möbius group of homographic transformations $\phi \rightarrow \frac{\alpha\phi + \beta}{\lambda\phi + \mu}$, depending on constants $\alpha, \beta, \lambda, \mu$ such that $\alpha\mu - \beta\lambda = 1$ (see e.g. [5] for the use in other contexts of expansions that systematically put forward the invariance properties under this group).

On the other hand, for $k \geq 1$, we obtain the relations

$$(k-1)\phi_y u_k - (k-1)\phi_x v_k + 2r_0 r_k = S_k^{(1)}, \quad (8a)$$

$$2r_0^2 v_k - (k-2)\phi_x w_k + 4v_0 r_0 r_k = S_k^{(2)}, \quad (8b)$$

$$-2r_0^2 u_k - (k-2)\phi_y w_k - 4u_0 r_0 r_k = S_k^{(3)}, \quad (8c)$$

$$-2r_0 u_0 u_k - 2r_0 v_0 v_k + r_0 w_k + (k^2 - 3k - 4)(\phi_x^2 + \phi_y^2) r_k = S_k^{(4)}, \quad (8d)$$

whose second members $S_k^{(i)}$, $i = 1, 2, 3, 4$, only involve functions u_j , v_j , w_j , r_j with $j < k$. More precisely

$$S_k^{(1)} = -u_{k-1,y} + v_{k-1,x} - \sum_{\substack{j+j'=k \\ j,j' \neq k}} r_j r_{j'}, \quad (9a)$$

$$S_k^{(2)} = -u_{k-2,t} - (k-2)\phi_t u_{k-1} + w_{k-1,x} - 2 \sum_{\substack{j+j'+j''=k \\ j,j',j'' \neq k}} v_j r_{j'} r_{j''}, \quad (9b)$$

$$S_k^{(3)} = -v_{k-2,t} - (k-2)\phi_t v_{k-1} + w_{k-1,y} + 2 \sum_{\substack{j+j'+j''=k \\ j,j',j'' \neq k}} u_j r_{j'} r_{j''}, \quad (9c)$$

$$\begin{aligned} S_k^{(4)} = & -(k-2)(\phi_{xx} + \phi_{yy})r_{k-1} - 2(k-2)(\phi_x r_{k-1,x} + \phi_y r_{k-1,y}) \\ & - r_{k-2,xx} - r_{k-2,yy} + 2C \sum_{\substack{j+j'+j''=k \\ j,j',j'' \neq k}} r_j r_{j'} r_{j''} \\ & - \sum_{\substack{j+j'=k \\ j,j' \neq k}} r_j w_{j'} + \sum_{\substack{j+j'+j''=k \\ j,j',j'' \neq k}} r_j (u_{j'} u_{j''} + v_{j'} v_{j''}). \end{aligned} \quad (9d)$$

In fact, Eqs.(8) define an algebraic linear system for u_k , v_k , w_k , r_k . Owing to (6), the associated determinant can be calculated and reduced to

$$Det(k) = -2B(\phi_x^2 + \phi_y^2)^3(k+1)(k-1)(k-2)(k-4). \quad (10a)$$

In this calculation and the following ones, both branches of r_0 in (6d) may indeed be treated simultaneously, which amounts to working with r^2 instead of r .

Note that $Det(k)$ vanishes at values k_r of k that are all integer, namely

$$k_r = -1, 1, 2, 4. \quad (10b)$$

Among these “resonance values”, $k_r = -1$ as usual reflects the arbitrariness of ϕ . For the other values of k_r , we have to check whether the equations are compatible *per se* or only for restricted ϕ satisfying some “consistency condition” [4]. In either case, since all roots of (10a) are simple, one of the functions u_{k_r} , v_{k_r} , w_{k_r} , r_{k_r} remains undetermined and correspondingly appears as arbitrary in the expansions (5a).

Let us add that, in connection with (3), it is also useful to write

$$u = -Bb + U, \quad v = Ba + V, \quad w = B\rho + W, \quad (11a)$$

with

$$a = \frac{r_x}{r}, \quad b = \frac{r_y}{r}, \quad \rho = r^2. \quad (11b)$$

Correspondingly, we have

$$u_k = -Bb_k + U_k, \quad v_k = Ba_k + V_k, \quad w_k = B\rho_k + W_k, \quad (11c)$$

where a_k, b_k, ρ_k and U_k, V_k, W_k are the coefficients involved in ϕ expansions like (5) for a, b, ρ and U, V, W . More precisely, a_k, b_k, ρ_k can be determined recursively from the $r_l, l \leq k$, by identification of powers of ϕ in (11b). The recurrence relations for U_k, V_k, W_k follow by similar operations in the equations for U, V, W obtained by inserting (11a) into (2). Alternatively, these relations may be deduced by substituting (11c) into (8), (9).

In fact, a splitting such as (11a) advantageously puts forward - and indeed yields for $U = V = W = 0$ - the form (11b) of the solutions associated with the $C^2 \equiv 1/B^2 = 1$ static configurations examined in [2]. Correspondingly, a part r_L that satisfies the Liouville equation (3) could have been separated off r , e.g. by writing $r = r_L + R$, with $R = 0$ for the solutions in [2]. However, for the present purpose, we only need to use U, V, W so as to cast the results in a sufficiently simple and structured form. In this respect, note that owing to (6) and to the fact that

$$a_0 = -\phi_x, \quad b_0 = -\phi_y, \quad \rho_0 = r_0^2 = B(\phi_x^2 + \phi_y^2), \quad (12)$$

we simply have $U_0 = V_0 = W_0 = 0$ for $k = 0$.

Let us now examine the cases $k = 1, 2, 3, 4$ successively.

2.1 The case $k = 1$ (resonance)

Owing to (9), the system (8) for $k = 1$ reduces to

$$2r_0r_1 = -u_{0y} + v_{0x}, \quad (13a)$$

$$2r_0^2v_1 + \phi_xw_1 + 4v_0r_0r_1 = \phi_tu_0 + w_{0x}, \quad (13b)$$

$$-2r_0^2u_1 + \phi_yw_1 - 4u_0r_0r_1 = \phi_tv_0 + w_{0y}, \quad (13c)$$

$$-2r_0u_0u_1 - 2r_0v_0v_1 + r_0w_1 - 6(\phi_x^2 + \phi_y^2)r_1 = (\phi_{xx} + \phi_{yy})r_0 + 2(\phi_xr_{0x} + \phi_yr_{0y}), \quad (13d)$$

with u_0, v_0, w_0, r_0 given by (6).

In fact, Eqs.(13b-d) have a vanishing determinant (therefore the whole system (13) too) and turn out to be compatible only if

$$(B^2 - 1)(\phi_{xx}\phi_y^2 - 2\phi_x\phi_y\phi_{xy} + \phi_{yy}\phi_x^2) = 0. \quad (14a)$$

Up to factors which are generally non vanishing, this condition is equivalent to (cf. (7))

$$C_1 \equiv (B^2 - 1)(\gamma_y - \gamma\gamma_x) = 0. \quad (14b)$$

Hence, the compatibility is ensured if $B^2 \equiv 1/C^2 = 1$ or/and if

$$\phi_{xx}\phi_y^2 - 2\phi_x\phi_y\phi_{xy} + \phi_{yy}\phi_x^2 = 0, \quad i.e., \quad \gamma_y - \gamma\gamma_x = 0. \quad (15a, b)$$

Note that (15a) is just the Bateman equation in two variables [6] whose determinantal form is

$$\det \begin{pmatrix} 0 & \phi_x & \phi_y \\ \phi_x & \phi_{xx} & \phi_{xy} \\ \phi_y & \phi_{yx} & \phi_{yy} \end{pmatrix} = 0.$$

At the end, if the “consistency condition” (14) - hereafter referred to as (C1) - is satisfied, then Eqs.(13b-d) determine two among the unknowns u_1 , v_1 , w_1 in terms of the third one which remains arbitrary. For further convenience, the result is here written as (cf. (11c))

$$u_1 = -Bb_1 + U_1, \quad v_1 = Ba_1 + V_1, \quad w_1 = B\rho_1 + W_1, \quad (16a)$$

with

$$a_1 = \phi_x \frac{r_1}{r_0} + \frac{r_{0x}}{r_0}, \quad b_1 = \phi_y \frac{r_1}{r_0} + \frac{r_{0y}}{r_0}, \quad \rho_1 = 2r_0 r_1, \quad (16b)$$

$$U_1 = \frac{\Gamma - \gamma \mathcal{H}}{2}, \quad V_1 = \frac{\mathcal{H}}{2}, \quad W_1 = B\phi_x[\gamma\Gamma - \mathcal{H}(1 + \gamma^2)], \quad (16c)$$

and, from (13a), (6a,b),

$$\rho_1 = 2r_0 r_1 = -B(\phi_{xx} + \phi_{yy}). \quad (16d)$$

In these expressions, r_0 has still to be replaced owing to (6d) while $\mathcal{H} \equiv \mathcal{H}(x, y, t)$ is an arbitrary function proportional to V_1 . On the other hand, Γ is the Möbius invariant

$$\Gamma = \frac{\phi_t}{\phi_x} \quad (17a)$$

which, owing to the required identity $(\phi_t)_y \equiv (\phi_y)_t$ (compatibility of the definitions (17a) and (7)), is such that

$$\Gamma_y - \gamma \Gamma_x \equiv \gamma_t - \Gamma \gamma_x. \quad (17b)$$

2.2 The case $k = 2$ (resonance)

In this case, Eqs.(8) and (9) give

$$\phi_y u_2 - \phi_x v_2 + 2r_0 r_2 = -u_{1y} + v_{1x} - r_1^2, \quad (18a)$$

$$2r_0^2 v_2 + 4v_0 r_0 r_2 = -u_{0t} + w_{1x} - 2(r_1^2 v_0 + 2v_1 r_0 r_1), \quad (18b)$$

$$-2r_0^2 u_2 - 4u_0 r_0 r_2 = -v_{0t} + w_{1y} + 2(r_1^2 u_0 + 2u_1 r_0 r_1), \quad (18c)$$

$$\begin{aligned} -2r_0 u_0 u_2 - 2r_0 v_0 v_2 + r_0 w_2 - 6(\phi_x^2 + \phi_y^2) r_2 = & -(r_{0xx} + r_{0yy}) \\ & + 6Cr_1^2 r_0 - w_1 r_1 + r_0(u_1^2 + v_1^2) + 2r_1(u_0 u_1 + v_0 v_1). \end{aligned} \quad (18d)$$

In fact, Eqs.(18a-c) have a vanishing determinant (therefore the whole system (18) too) and turn out to be compatible only if

$$2B(\phi_x^2 + \phi_y^2)S_2^{(1)} + \phi_x S_2^{(2)} + \phi_y S_2^{(3)} = 0 \quad (19a)$$

where the $S_2^{(i)}$, $i = 1, 2, 3$, stand for their second members.

This new condition is hereafter referred to as (C2). Owing to (6), (7), (16) and (17), and up to global factors which are generally non vanishing, it simply reduces to

$$C_2 \equiv \Gamma_y - \gamma\Gamma_x - \mathcal{H}(\gamma_y - \gamma\gamma_x) = 0, \quad (19b)$$

where $\Gamma_y - \gamma\Gamma_x$ may be replaced by $\gamma_t - \Gamma\gamma_x$ just as well (cf. (17b)). In the general case, with an arbitrary \mathcal{H} , this requires $\gamma_y = \gamma\gamma_x$ (Eq.(15b)), *and*

$$\Gamma_y = \gamma\Gamma_x, \text{ or equivalently, } \gamma_t = \Gamma\gamma_x. \quad (20a, b)$$

Note that, in terms of the derivatives of ϕ , the latter condition also reads as

$$2\phi_x\phi_t(\phi_{yt}\phi_x - \phi_{xt}\phi_y) + \phi_t(\phi_{xx}\phi_y^2 - \phi_{yy}\phi_x^2) = 0$$

and amounts to the determinantal three variable equation [7]

$$\det \begin{pmatrix} 0 & \phi_x & \phi_y & \phi_t \\ \phi_x & \phi_{xx} & \phi_{xy} & \phi_{xt} \\ \phi_y & \phi_{yx} & \phi_{yy} & \phi_{yt} \\ \phi_t & \phi_{tx} & \phi_{ty} & \phi_{tt} \end{pmatrix} = 0,$$

restricted by (15a).

At the end, provided that (19) holds, Eqs.(18a-d) yield for instance u_2 , v_2 , w_2 in terms of r_2 taken arbitrary and assumed, without loss of generality, to be independent of Γ and \mathcal{H} . The result, when written in the form (11c), involves

$$a_2 = 2\frac{r_2}{r_0}\phi_x + \frac{r_{1x}}{r_0} - \frac{r_1}{r_0}a_1, \quad b_2 = 2\frac{r_2}{r_0}\phi_y + \frac{r_{1y}}{r_0} - \frac{r_1}{r_0}b_1, \\ \rho_2 = 2r_0r_2 + r_1^2. \quad (21a)$$

Moreover, we find that

$$U_2\phi_x = -\frac{(\Gamma - \gamma\mathcal{H})_y}{2\gamma}, \quad V_2\phi_x = -\frac{\mathcal{H}_x}{2}, \quad W_2 = (B^2 - 1)W_2^B + W_2^{nB}, \quad (21b, c, d)$$

with

$$W_2^B = a_1^2 + b_1^2 + 2a_0a_2 + 2b_0b_2 - C\rho_2, \quad (21e)$$

$$W_2^{nB} = -2B(U_{1y} - V_{1x}) + U_1^2 + V_1^2 - 2B(U_1b_1 - V_1a_1). \quad (21f)$$

In all these expressions, r_0 , a_0 , b_0 , a_1 , b_1 , U_1 , V_1 , r_1 have to be explicated according to (6d), (12) and (16b-d). In fact, owing to the factor $(B^2 - 1)$, the

Γ - and \mathcal{H} -independent part W_2^B needs to be evaluated for $\gamma_y = \gamma\gamma_x$ only (the constraint (C1) requires that every power or derivative of $(B^2 - 1)(\gamma_y - \gamma\gamma_x)$ cancels). This simply yields

$$W_2^B = -6Cr_0r_2 - 6Cr_1^2 + \frac{r_{0xx} + r_{0yy}}{r_0}. \quad (21g)$$

Similarly, owing to (C2), the combinations $(\Gamma_y - \gamma\Gamma_x)$ and $(\gamma_t - \Gamma\gamma_x)$ may be replaced by $\mathcal{H}(\gamma_y - \gamma\gamma_x)$.

2.3 The case $k = 3$

For $k = 3$, the system (8) becomes

$$2\phi_y u_3 - 2\phi_x v_3 + 2r_0 r_3 = S_3^{(1)}, \quad (22a)$$

$$2r_0^2 v_3 - \phi_x w_3 + 4v_0 r_0 r_3 = S_3^{(2)}, \quad (22b)$$

$$-2r_0^2 u_3 - \phi_y w_3 - 4u_0 r_0 r_3 = S_3^{(3)}, \quad (22c)$$

$$-2r_0 u_0 u_3 - 2r_0 v_0 v_3 + r_0 w_3 - 4(\phi_x^2 + \phi_y^2)r_3 = S_3^{(4)}, \quad (22d)$$

and has a non vanishing determinant, cf.(10). Hence, this system admits a unique solution which may be readily expressed in terms of the $S_3^{(i)}$ and thus, owing to (9), in terms of the $u_k, v_k, w_k, r_k, k \leq 2$, determined previously. In correspondance with (11c), (16) and (21), splittings such as

$$u_3 = -Bb_3 + U_3, \quad v_3 = Ba_3 + V_3, \quad w_3 = B\rho_3 + W_3, \quad (23a)$$

may also be considered, together with structures in the U_3, V_3, W_3 that have well defined behaviours at relevant limits such as $B^2 = 1$ or/and $\Gamma = 0$ (static limit) or/and $\mathcal{H} = 0$. In particular, it is useful to write

$$U_3 = (B^2 - 1)U_3^B + U_3^{nB}, \quad V_3 = (B^2 - 1)V_3^B + V_3^{nB}, \quad W_3 = (B^2 - 1)W_3^B + W_3^{nB}, \quad (23b)$$

where all contributions independent of Γ and \mathcal{H} are gathered in the U_3^B, V_3^B, W_3^B . Note that these B -indexed parts, already put forward in Eq.(21d) and such that $U_k^B = V_k^B = 0$ for $k \leq 2$, $W_k^B = 0$ for $k \leq 1$, are at the same time multiplied by an overall factor $(B^2 - 1)$. Owing to their definition, they are found again in an analysis of static forms of Eqs.(2) which for the U, V, W read as (cf. (11a,b))

$$U_y - V_x = B\mathcal{L}(r), \quad W_x = 2Vr^2, \quad W_y = -2Ur^2, \\ \mathcal{L}(r) = (B^2 - 1)(a^2 + b^2 - Cr^2) - W + U^2 + V^2 + 2B(Va - Ub), \quad (24)$$

with $\mathcal{L}(r)$ given by (3) or $\mathcal{L}(r) \equiv a_x + b_y - C\rho$.

In any case, the expressions of $a_3, b_3, \rho_3 = 2(r_0 r_3 + r_1 r_2)$, U_3, V_3, W_3, r_3 become quite lengthy - and therefore are not explicated here - when everything is replaced in terms of the arbitraries, besides basic x -derivatives of ϕ and Möbius invariants. As Eq.(21g), they involve the Schwartzian $S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2} \frac{\phi_{xx}^2}{\phi_x^2}$ whose derivatives S_y, S_t can be expressed in terms of S_x thanks to (7) and (17a).

2.4 The case $k = 4$ (resonance)

The corresponding Eqs.(8) become

$$3\phi_y u_4 - 3\phi_x v_4 + 2r_0 r_4 = S_4^{(1)}, \quad (25a)$$

$$2r_0^2 v_4 - 2\phi_x w_4 + 4v_0 r_0 r_4 = S_4^{(2)}, \quad (25b)$$

$$-2r_0^2 u_4 - 2\phi_y w_4 - 4u_0 r_0 r_4 = S_4^{(3)}, \quad (25c)$$

$$-2r_0 u_0 u_4 - 2r_0 v_0 v_4 + r_0 w_4 = S_4^{(4)}, \quad (25d)$$

where the $S_4^{(j)}$, $j = 1, 2, 3, 4$, can be expressed thanks to (9) in terms of the u_k, v_k, w_k, r_k , $k \leq 3$, obtained previously, or directly in terms of the $S_k^{(j)}$, $k \leq 3$. In fact, Eqs.(25) have a vanishing determinant (cf.(10)) and are compatible only if

$$C_4 \equiv 2r_0^2 S_4^{(1)} + \phi_x S_4^{(2)} + \phi_y S_4^{(3)} + 2C r_0 S_4^{(4)} = 0. \quad (26)$$

The analysis of this last condition - referred to as (C4) below - *a priori* appears quite involved. Nevertheless, large simplifications occur which we have also checked with the help of computer programs (MATHEMATICA, REDUCE) allowing symbolic manipulations. In this respect, it is advantageous to take into account the structures put forward in (11c), (21d) and (23), combined with the preceding constraints (C1), (C2). At the end we obtain

$$C_4 \equiv (B^2 - 1)C_4^B + C_4^{nB} = 0, \quad (27a)$$

with

$$C_4^B \equiv \frac{2}{3(1 + \gamma^2)} (\delta_\eta)^2 W_2^B, \quad (27b)$$

$$C_4^{nB} \equiv -\delta_\eta \delta_T \mathcal{H}, \text{ or, owing to (C2), } C_4^{nB} \equiv -\delta_T \delta_\eta \mathcal{H} + (\delta_\eta \mathcal{H})^2, \quad (27c)$$

$$\delta_\eta \equiv \partial_y - \gamma \partial_x, \quad \delta_T \equiv \partial_t - \Gamma \partial_x - \mathcal{H} \delta_\eta. \quad (28)$$

Again, the Γ - and \mathcal{H} -independent part C_4^B , with W_2^B given by (21e), may be evaluated for $\gamma_y = \gamma \gamma_x$ (in such conditions $(B^2 - 1)(\delta_\eta)^2 = (B^2 - 1)(\partial_{yy} - 2\gamma \partial_{xy} + \gamma^2 \partial_{xx})$).

For further discussions, let us add that

(i) alternatively, C_4 may be split into

$$C_4 \equiv C_4^S + C_4^T, \quad (29a)$$

with

$$C_4^S \equiv (B^2 - 1)C_4^B + \mathcal{H}(\delta_\eta)^2 \mathcal{H} + (\delta_\eta \mathcal{H})^2, \quad C_4^T \equiv -(\partial_t - \Gamma \partial_x) \delta_\eta \mathcal{H}. \quad (29b, c)$$

Obviously, C_4^T cancels in the static limit ($\partial_t \rightarrow 0$, $\Gamma \rightarrow 0$) and for $\mathcal{H} \rightarrow 0$, while in such limits, C_4^S reduces to $(B^2 - 1)C_4^B$ (cf. the remarks in Section 2.3 concerning the B indexed parts).

(ii) the first two constraints (C1), (C2) (cf. (14b), (19b)) may also be simply rewritten in terms of the just introduced operators δ_η , δ_T as

$$C_1 \equiv (B^2 - 1)\delta_\eta\gamma = 0, \quad (30a)$$

and (cf.(17b))

$$C_2 \equiv \delta_\eta\Gamma - \mathcal{H}\delta_\eta\gamma = \delta_T\gamma = 0. \quad (30b)$$

3 Discussion and conclusion

It is clear on expressions (27), (29) that C_4 only vanishes for specific \mathcal{H} and W_2^B - i.e. specific \mathcal{H} and r_2 owing to (21e,g) - which contradicts the assumptions made in steps $k = 1, 2$ on the arbitrariness of these functions. Therefore, although the analysis at $k = 1, 2$ might imply the existence of some “conditional Painlevé property” for Eqs.(2a-d) (cf.[4]), the constraint at $k = 4$ shows that generally no such property holds at all. According to the point of view developped in [4], this suggests that Eqs.(1), (2) are *not* integrable.

These results agree with others that appeared during the completion of the present work. Namely, in [8] two ODE reductions of (1) are shown to possess the Painlevé property, whereas another one, associated with rotational invariance, does not (in fact, owing to the above study, the first two ODE may be easily identified with static and similarity reductions of the integrable NLSE Eq.(4)).

Precisely, the analysis of Section 2 provides us with tools for investigating which reductions of Eqs.(1), (2) might be integrable, i.e. are such that all conditions (C1), (C2), (C4) are identically satisfied. Let us here merely mention that

i) In the static limit obtained by dropping all time dependences, a 2+0 reduction of (2) is got for which all the required conditions are generally no more satisfied. As a direct WTC analysis also shows, we have in this case $C_4 \equiv C_4^S$ (cf.(29)), which does not vanish for any \mathcal{H} . This suggests that such a reduction, indeed equivalent to two coupled second order equations for r , w as well as to Eqs.(24), is no more integrable.

ii) If we furthermore restrict ourselves in case (i) to values $C^2 \equiv 1/B^2 = 1$ and to solutions built with $\mathcal{H} = 0$, then all the conditions (C1), (C2), (C4) become satisfied whatever $\phi(x, y)$ is. At the same time, the selected solutions are such that U_k , V_k , W_k vanish for $k \leq 2$ (cf. Eqs.(12), (16c) and (21b-f) with $\Gamma = \mathcal{H} = 0$), and indeed for any k owing to the recursion laws. Therefore, we have $U = V = W = 0$ for such solutions, which, owing to (24), are clearly associated with the integrable Liouville equation $\mathcal{L}(r) = 0$. In fact, the analysis in Section 2 reduces in this case to that of the 2+0 Eq.(3) and, as it should be, the choice $\mathcal{H} = 0$ implies a diminution of the arbitrariness in comparison with the case of the full Eqs.(2) (Eq.(3) has only two resonances at $k = -1, 2$).

iii) On the other hand, if all the involved quantities - among which ϕ , γ , Γ , \mathcal{H} and W_2^B - only depend on x , y through the combination $\xi = x + \gamma_c y$, γ_c constant, then $\delta_\eta \gamma = 0$ (indeed $\gamma = \gamma_c$) and $\delta_\eta \Gamma = \delta_\eta \mathcal{H} = \delta_\eta W_2^B = 0$. Therefore, all the resonance conditions are fulfilled for any $C \equiv 1/B$ and $\phi(\xi, t)$ (cf.(30), (27)). In this case, the analysis of Section 2 only involves quantities which are related (cf.(4)) to the phase and modulus of complex functions $\Psi(\xi, t)$ satisfying the integrable 1+1 NLSE.

Independently of the remarks (i)(ii)(iii) above, we may also expect to take advantage of the above analysis for obtaining solutions of the equations. In this respect, it has been emphasized by several authors (see e.g.[9]) that the use of truncated series is often fruitful, not only in integrable, but also in non integrable cases.

This problem and those arising with the use of (C1), (C2), (C4) for disclosing the integrability properties of different reductions of (2) - among which those found in connection with group symmetry properties - are examined elsewhere [10].

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